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APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD UNIVERSITY
CALIFORNIA

ACCOMMODATION OF SECOND-CLASS TRAFFIC

By

DONALD P. GAVER, JR.

TECHNICAL REPORT NO. 74

February 26, 1962

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1. General Description of the Traffic Problem

Various situations seem to give rise to traffic problems of the following sort: a service facility exists for a certain primary function, but is not always busy performing it--gaps exist between periods of primary function performance. The gaps are thus available for some secondary tasks, the performance of which must not disturb the facility's primary activity; the traffic composed of such tasks may be called second-class. Query: what grade of service is the lot of the secondary tasks; e.g., how long does such a task wait, how many tasks are likely to be performed in a given time period, etc.? Further, we may ask about sensible ways of scheduling the order of secondary tasks, for sometimes this is possible.

Examples of traffic problems of the kind mentioned are these:

(a) The problem of crossing a railroad track between trains with a slow-moving vehicle or convoy of trucks.

(b) The problem of gaining access to a major "through" street or highway from a sidestreet, when the crossing has no policeman or traffic light.

(c) The problem of preventively maintaining a computer, radar, machine tool, or vehicle (e.g. fire engine or police car) between occasions of such an object's usage for its primary purpose.

(d) The problem of message transmission when bursts of noise of varying

durations occur occasionally.

(e)   The problem of producing special factory orders between production runs, when the same equipment is used for both, and the special orders are of low priority. Other examples will suggest themselves, but in all of those just mentioned some facility is used principally for a primary function, but one that occurs sporadically. Thus opportunities exist for performance of low-priority tasks in what would otherwise be idle periods. We investigate the quality of the resulting service to the tasks.

Very simple probabilistic assumptions have been made in the discussion to follow. These have been used to answer simple questions, the answers to which are mainly also simple if sometimes a little surprising. Sometimes, too, the simple formulations and simple questions lead to complex-appearing answers. In particular, we have been able to discuss the waiting line, or queue, of low-priority tasks that may form before the facility when tasks appear "at random" (Poisson-wise). Even the expected length of this queue under long-run conditions has a complicated formula. Investigation of this formula undoubtedly requires numerical computation and tables, and these are not included here. It is to be expected that more realistic, and probably less simple, basic assumptions than have been made here will lead to even more involved mathematical results.

Another restriction of the present treatment is that of neglecting the inference and scheduling problems of the administrator responsible for tasks. He will very likely not know the length of idle periods precisely, and he may want to dictate the order in which tasks are

2

performed. Problems that arise from these sources will be solved where possible in future work.

2.   Problem Formulation

It is clear that the quality of low-priority task service, as reflected in waiting times, etc., depends upon the pattern of the facility's primary function usage. From the view point of the tasks, this pattern is an alternating sequence of <u>blocks</u>, during which the facility is performing its primary function, and <u>gaps</u> or idle periods, during which it is potentially available for tasks service. Formally, the <u>facility is blocked</u> during the time intervals

$$(t_n', t_n''], \qquad\qquad n = 1, 2, \dots \ ;$$

t_n' is the instant at which the $n\underline{th}$ blocking period begins, and t_n'' is the instant at which this same blocking period (or block, for short) ends. For all n

$$t_n' < t_n'' < t_{n+1}' < t_{n+1}''$$

and the duration of the $n\underline{th}$ block is

$$B_n = t_n'' - t_n' \qquad\qquad (2.1)$$

The facility is <u>free</u> or <u>idle</u>, or a <u>gap is present</u> during the intervals

$$(t_{n-1}'', t_n'] \qquad\qquad n = 1, 2, \dots \ ;$$

3

$t_0'' = 0$ may be taken as the initial instant. The duration of the nth gap is

$$G_n = t_n' - t_{n-1}'', \qquad\qquad n = 1, 2, \ldots \ . \qquad (2.2)$$

During the gaps the tasks may use the facility. The delays experienced by the latter will depend upon the times at which they appear, their durations, and the rules adopted for scheduling tasks through the gaps. Of course if the tasks require facility time while blocks are in progress they will be forced to wait their turn, and a queueing phenomenon will occur. This problem will be discussed.

3.   A Renewal Model for Primary Function Utilization Pattern.

It is possible to introduce various sets of assumptions about the pattern of facility usage, both for the high priority principal function and the low priority tasks. The following model is probably the simplest possible; it is considered because it is relatively plausible, easy to work with and provides some interesting results, and because models like it have been introduced by others.

Suppose that a history of high priority facility usage is a two-state renewal process. Thus we assume

(a) $\{B_n\}$, the sequence of block durations, is a sequence of independent random variables with common d.f. $\beta(x)$.

(b) $\{G_n\}$, the sequence of gap durations, is also a sequence of independent r.v., worth common d.f. $\gamma(x)$.

(c)   The sequences $\{B_n\}$ and $\{G_n\}$ are mutually independent.

4

Tasks, too, may be conceived of as appearing at the facility in accordance with a random process, but this will not be made explicit now. We shall, however, suppose that task durations are a sequence, $\{L_n\}$, of independent random variables, with common d.f. $b(x)$; the sequence $\{L_n\}$ is independent of $\{B_n\}$ and $\{G_n\}$ also. Where no confusion results the subscript n will be dropped.

The obstacles to task service presented by the principal, or high-priority, function may be summarized in the high-priority state of the facility, henceforth the state for short; this is the two-dimensional r.v. $[C(t), T(t)]$, where

and

$$\left.\begin{array}{l} C(t) = 1 \\ \\ T(t) = t_n'' - t \end{array}\right\} \text{if } t\epsilon(t_n', t_n''] \text{ for some } n = 1, 2, \ldots \quad (3.1)$$

otherwise

$$\left.\begin{array}{l} C(t) = 0 \\ \\ T(t) = t_n' - t \end{array}\right\} \text{if } t\epsilon(t_{n-1}'', t_n'] \quad n = 1, 2, \ldots \quad (3.2)$$

In words, $C(t)$ is unity (zero) if the facility is blocked (a gap is present) at t, and $T(t)$ is the waiting time until the end of the current block (gap).

If initially a gap is just beginning, $C(0) = 0$, $T(0) = t_o'' = 0$, the state at time t may be obtained by noting that, for example, $C(t) = 0$, $0 < T(t) \leq x$ if any of the following mutually exclusive events occur:

5

(1)   The first block begins after time t,   but before t+ x; the probability is

$$P[t < I_1 \le t + x] = \gamma(t + x) - \gamma(t) \ ; \tag{3.3}$$

in general (n)   the $n\underline{th}$ (n = 1, 2, ...) block terminates before t, and the $n + 1\underline{st}$ block begins after t but before t + x; the probability is

$$P\left[\sum_{i=1}^{n} (I_i + B_i) < t < \sum_{i=1}^{n} (I_i + B_i) + I_{n+1} \le t + x \right]$$

$$\tag{3.4}$$

$$= \int_o^t \left[\gamma(t+x-y) - \gamma(t-y) \right] \gamma_*^{n^*} \beta^{n^*}(dy)$$

The sum of the probabilities given

$$P\left[C(t) = 0, \ 0 < T(t) \le x \ \middle| \ C(0) = 0, \ t_o'' = 0 \right]$$

$$= \int_o^t \left[\gamma(t+x-y) - \gamma(t-y) \right] H_1(dy) \tag{3.5}$$

where

$$H_1(y) = \sum_{n=o}^{\infty} \gamma_*^{n^*} \beta^{n^*}(y) \tag{3.6}$$

6

and

$$\gamma_*^{o*} \, \beta^{o*}(y) = \begin{cases} 1 & y \geq 0 \\ \\ 0 & y < 0 \end{cases}$$

The unitary d.f.

A similar decomposition results in a comparable expression for

$$P[C(t) = 1, \; 0 < T(t) \leq x \, | \, C(0) = 0, \; t_o'' = 0]$$

$$= \int_o^t [\beta(t+x-y) - \beta(t-y)] \; H_2\{dy\} \qquad (3.7)$$

where here $H_2(y) = \gamma^* H_1(y)$.

The assumptions (a), (b), and (c) provide that the sequence $\{I_n + B_n\}$ is a renewal process. By use of well-known asymptotic results, cf. Smith [1], we may write down probabilities relating to the state as $t \to \infty$, i.e. in the long run. Thus, apply Theorem 1 of Smith [1], making the following identification in (3.5):

$$\Psi(t) = \gamma(t+x) - \gamma(t) = [1 - \gamma(t)] - [1 - \gamma(t+x)] \; .$$

$$(3.8)$$

$$H_2(t) = H_1(t)$$

Henceforth, assume unless stated otherwise that all required moments of γ and β are finite and that at least one of these d.f.s possesses an absolutely continuous component. Then we verify the conditions for Smith's Theorem:

7

(1) $\psi(t)$, being a probability, is bounded for $t \geq 0$;

(2) $$\int_0^\infty \psi(t)dt = \int_0^\infty [1-\gamma(t)]dt - \int_0^\infty [1-\gamma(t+x)]dt$$

is certainly finite if $E(G_n)$ is finite, and the latter is assumed;

(3) $$\lim_{t \to \infty} [1-\gamma(t)] = 0, \quad \text{hence} \quad \psi(t) \to 0;$$

(4) $\gamma * \beta$ has an absolutely continuous component;

(5) $E(G_n + B_n) = E(G_n) + E(B_n)$ is finite, by assumption.

It follows that

$$\lim_{t \to \infty} P[C(t) = 0, \ 0 < T \leq x \mid C(0) = 0, \ t_0'' = 0] =$$

(3.9)

$$P[C = 0, \ 0 < T \leq x] = \frac{E(G)}{E(G)+E(B)} \left\{ \frac{\int_0^x [1-\gamma(y)]dy}{E(G)} \right\}$$

and that

$$P[C = 1, \ 0 < T \leq x] = \frac{E(B)}{E(G)+E(B)} \left\{ \frac{\int_0^x [1-\beta(y)]dy}{E(B)} \right\}$$ (3.10)

where $[C, T]$ is the long-run state of the facility.

8

4.   Figures of Merit for Task Service

The description given of h.p. state makes possible a study of low-priority task delays. Answers to a variety of questions are almost immediate, and will perhaps be useful even if the model is not precisely accurate.

(1)   If a task suddenly makes application for service at a time when the long-run d.f. of system state applies, then the probability that it encounters a block is, of course,

$$\frac{E(B)}{E(G) + E(B)} \ .$$
(4.0)

It must not be assumed that the complementary probability is that of immediate service for the task. Suppose that the task is indivisible, i.e. it cannot be done in segments. For example, one cannot drive a truck or car across a railroad track in segments, with a block (train) separating the segments, although this seems sometimes to be attempted. On the other hand, message transmission can probably be accomplished in segments, and such tasks might be called divisible. Our entire interest here will be in indivisible tasks. It will become obvious that divisible tasks would suffer fewer delays.

Then the probability that a task enconcounters an inadequate gap, i.e. one shorter than its own duration, is

$$P[C = 0, \ L > T] = \frac{E(G)}{E(G)+E(B)} \left\{ \frac{\int_0^\infty [1-b(x)][1-\gamma(x)]dx}{E(G)} \right\} \ .$$
(4.1)

9

The probability that a task encounters an adequate gap is

$$P[C = 0, \; L < T] = \frac{E(G)}{E(G)+E(B)} \left\{ \frac{\int_0^\infty b(x)[1-\gamma(x)]dx}{E(G)} \right\} . \quad (4.2)$$

Note that (4.1) may be presented in the alternate form

$$\frac{E[Min(G, \; L)]}{E(G) \; + \; E(B)}$$

Unlike (4.0), this probability depends in some detail on the form of the
d.f.' γ and b-- not just on their means. The event that a task is
either initially blocked or encounters an inadequate gap, i.e. the
probability that the task must wait is

$$\frac{E(B) \; + \; E[Min(G, \; L)]}{E(G) \; + \; E(B)} . \quad (4.3)$$

Remark from (3.9) that, given that a task appears during a gap, the
remaining duration of the gap has d.f.

$$P[T \leq x] = \frac{\int_0^x [1-\gamma(y)]dy}{E(G)} ; \quad (4.4)$$

the resulting d.f. has a monotone decreasing density, whatever the d.f.
γ, and a mean value

$$E(T) = \frac{1}{2} \; \frac{E(G^2)}{E(G)} = \frac{1}{2} \left\{ \frac{Var(G) \; + \; E^2(G)}{E(G)} \right\} \quad (4.5)$$

10

Clearly, if $Var(G)$ is small, then the expected remaining time is not much bigger than $\frac{1}{2} E(G)$, and the chance of having to wait will tend to be higher here than if the task had applied at the beginning of a gap. This situation conforms to the usual intuitive expectation. On the other hand, suppose $\gamma(x)$ is a _mixture of exponentials_:

$$1 - \gamma(x) = \int_0^\infty e^{-\alpha x} V\{d\alpha\} \quad 0 \leq x < \infty \tag{4.6}$$

where V is a d.f. on the rate parameter α. Note that

$$E(G) = \int_0^\infty [1-\gamma(x)]dx = \int_0^\infty \frac{V\{d\alpha\}}{\alpha} \tag{4.7}$$

provided the latter exists. If a task appears during a gap, then the remaining gap duration has d.f.

$$P[T > x] = \frac{\int_x^\infty [1-\gamma(y)]dy}{E(G)} = \frac{\int_0^\infty \frac{e^{-\alpha x}}{\alpha} V\{d\alpha\}}{\int_0^\infty \frac{V\{d\alpha\}}{\alpha}} \tag{4.8}$$

and hence, if the integrals exist,

$$E(T) = \frac{\int_0^\infty \frac{1}{\alpha^2} V\{d\alpha\}}{\int_0^\infty \frac{1}{\alpha} V\{d\alpha\}} = \int_0^\infty \frac{1}{\alpha} \tilde{V}\{d\alpha\} \tag{4.9}$$

11

and $\quad \widetilde{V}(\alpha) = \int_0^\alpha \frac{V(d\eta)}{\eta} \left[\int_0^\infty \frac{V(d\eta)}{\eta} \right]^{-1}$ It is clear that the measure $\widetilde{V}$

assigns relatively more weight to small values of α than does V,

and hence $E(T) \geq E(G)$. The implication is that if the gap d.f. has

the above structure, then the chance of having to wait for a time $\geq t$

may be larger here than if the task had applied at the beginning of a

gap randomly selected from $\gamma(x)$. The reason for this is, of course,

that if the facility is approached after it has been in operation for

some time, and a gap is observed, then the gap observed is not a sample

from $\gamma(x)$-- roughly speaking, long gaps have a better chance of being

observed. So pronounced is this effect for distributions of the class

(3.16) that, for every $x > 0$, $P[T > x] \geq P[G > x]$, where G is the

duration of an arbitrary gap randomly selected from γ. Thus, whatever

our task length, the chance of getting it through without a wait is

higher if it encounters a gap in the long-run than if it merely

confronts one arbitrarily selected from γ.

(2)   Suppose a task arrives and must wait. Let N denote the number

of gaps that pass by until the first one appears sufficient to

accommodate the task. The d.f. of N is derived as follows: temporarily

fix $L = x$; then the probability that $N > j$ is, by independence,

$$P[N > j \mid L = x] = \gamma^j(x) \qquad\qquad j = 0, 1, 2, \ldots \ ;$$

removal of the condition on L gives

12

$$P[N > j] = \int_0^\infty \gamma^j(x) \, b\{dx\} \quad . \tag{4.10}$$

It follows that

$$P[N=j] = P[N > j-1] - P[N > j] = \int_0^\infty \gamma^{j-1}(x) \, [1-\gamma(x)] \, b\{dx\} \tag{4.11}$$

and that

$$E[N] = \int_0^\infty \frac{b\{dx\}}{1-\gamma(x)} \quad . \tag{4.12}$$

From these formulas some interesting qualitative results emerge:

(a) $P[N=j]$ is from (4.10) a decreasing function of j, so the modal value of N is unity, the smallest value possible.

(b)   Suppose $b(x) = \gamma(x)$, or task-length d.f. is exactly the same as that of gap-length. Although intuition may suggest that such a setup is compatible--tasks "should have" about an even chance of fitting into gaps--we see from (4.12) that $E(N)$, the expected number of gaps that must elapse before accommodation, is infinite! In this case the actual d.f. of N is available; from (4.10) we get

$$P(N > j) = \int_0^\infty \gamma^j(x) \, \gamma\{dx\} = \frac{1}{j+1} \qquad j = 1, 2, \ldots \quad , \tag{4.13}$$

which adds the further information that the median value of N is also unity. It is apparent that the d.f. of N is one of considerable positive skewness, observations from which are likely to be subject

13

to practical misinterpretation. The same phenomena occurs in connection with the <u>theory of records</u>, see Chandler [2].

This same behavior -- an infinite $E(N)$ -- will result if for every x, $P(G \leq x) > P(L \leq x)$, or $\gamma(x) > b(x)$, for

$$E(N) = \int_0^\infty \frac{b\{dx\}}{1-\gamma(x)} > \int_0^\infty \frac{b\{dx\}}{1-b(x)} = \infty \qquad (4.14)$$

(c) Suppose gap lengths have the <u>exponential</u> d.f.: $1 - \gamma(x) = e^{-\nu x}$. Then it follows immediately from (4.12) that $E(N) = \hat{b}(-\nu)$, if the latter exists, where $\hat{b}(s)$ is the Laplace-Stieltjes transform of the task length d.f. Note that by making use of a probability mixture of exponential d.f.'s for tasks we can construct a task length d.f. with an arbitrarily small positive mean, but one for which $E(N)$ is infinite. We need only make our task length d.f. concentrate at 0 (be unitary) with probability p, and to have the gap length d.f., $1-e^{-\nu t}$, with probability $q = 1-p$. For $q > 0$ the expression for $E(N)$ obviously diverges, and yet since expected task length is q/ν we can make the tasks arbitrarily short on the average by making q small. This sort of example is probably artificial, but it demonstrates that a consideration of the simplest averages may not be sufficient in this problem, as in many others.

(3) Suppose k tasks are waiting, and suppose all are equally able to use the facility when a gap appears. Let $N_{[k]}$ be the number of gaps that must occur in order to find one sufficient for the <u>shortest</u> task. To find the d.f. of $N_{[k]}$, fix $L_{[k]} = x$; then the next j gaps are too short with probability $\gamma^j(x)$, so, unconditionally,

14

$$P[N > j] = \int_0^\infty \gamma^j(x) \; k[1-b(x)]^{k-1} \; b(dx) \quad . \qquad (4.15)$$

(a) Again as a special case take $\gamma = b$. Then

$$P[N_{[k]} > j \,|\, k] = k \int_0^\infty b^j (1-b)^{k-1} db = kB(j+1,k) = \left[\binom{j+k}{j} \right]^{-1} \qquad (4.16)$$

where $B(.,.)$ is the ordinary Beta function. In particular, we have for $k = 2$

$$P[N_{[2]} > j \,|\, 2] = \frac{2}{(j+1)(j+2)} \qquad (4.17)$$

and it is clear that $E(N_{[2]})$ is finite. To find it explicitly, rearrange the series as follows

$$E(N_{[2]}) = 2 \left[\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots \right]$$

$$\qquad (4.18)$$

$$= 2 \left[(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots \right] = 2$$

Thus if two tasks are waiting, an average of only two gaps goes by until one can get through. Of course the average number for the other is again infinite.

The above indicates the possible evils of an inflexible "first-come, first-served" policy for tasks. To elaborate, suppose that two tasks are waiting, and that the one in first place has task length x, while the one in second place has length y, where $y < x$. We ask

15

how many gaps long enough to accommodate y go by before one long enough for x appears. Let M be the number of missed opportunities for the second task, i.e. the number of gaps larger than y but smaller than x that the second task may not take because of the presence of the first task. The probability of a missed opportunity is seen to be

$$\frac{\gamma(x) - \gamma(y)}{1 - \gamma(y)} \; ,$$

so the chance of m or more before the first task departs is

$$P[M \geq m \,|\, x,y] = \left[\frac{\gamma(x) - \gamma(y)}{1 - \gamma(y)}\right]^{m} \qquad m = 0, \; 1, \; 2, \; \ldots \quad (4.19)$$

The probability element associated with x, y is

$$2b\{dx\} \; b\{dy\} \qquad 0 < y < x < \infty$$

(we are assuming b and γ have densities for convenience), so

$$P[M \geq m] = 2 \int_{0}^{\infty} b\{dy\} \int_{y}^{\infty} \frac{\gamma(x) - \gamma(y)}{1 - \gamma(y)}^{\,m} b\{dx\} \; . \qquad (4.20)$$

Again let b = γ; we integrate directly and get

$$P[M \geq m] = \frac{1}{m+1} \qquad\qquad m = 0, \; 1, \; 2, \; \ldots \qquad (4.21)$$

16

Notice that this is entirely equivalent to the distribution of N, the number of the gap, following his arrival, that accommodates the first task. In fact, the d.f. of M + 1 and N are seen to be identical, by reference to (4.13). Thus in this particular special case, the second task in line either misses no opportunities, an event of probability $\frac{1}{2}$, or, with probability $\frac{1}{2}$, must futilely watch several (infinitely many, on the average) perfectly useable gaps go by.

Examine now the following special

Example. Put $\gamma(x) = 1 - e^{-\nu x}$, and $b(x) = 1 - e^{-\mu x}$.

We already know that if $\gamma \geq b$ for all x, implied by $\nu \geq \mu$, then the expected number of gaps that must go by before a suitable one occurs is infinite. We may, however, characterize the d.f. of M above, and calculate the expectation in the present case; (4.20) becomes

$$P[M \geq m] = 2 \int_0^\infty e^{-\mu y} \mu dy \int_y^\infty \left[\frac{e^{-\nu y} - e^{-\lambda x}}{e^{-\nu y}} \right]^m e^{-\mu x} \mu dx$$

Putting $z = e^{-\nu(x-y)}$, and performing the integration on y we immediately get

$$P[M \geq m] = \frac{\mu}{\nu} \int_0^1 (1-z)^m z^{\frac{m}{\nu} - 1} dz = \frac{\mu}{\nu} B(m+1, \frac{\mu}{\nu}) \qquad (4.22)$$

It can be seen that if $\mu > \nu$ at least the first moment exists, so let us sum under the integral in (4.22) and then integrate, a procedure validated by Fubini; we get

17

$$E(M+1) = \frac{1}{1-(v/\mu)} = \frac{E(\text{gap duration})}{E(\text{gap duration}) - (E(\text{task duration}))} \qquad (4.23)$$

If the expected task length is 80% of the expected gap length an average

of 5 missed opportunities (including the gap that accommodates the task

ahead of him in line) will be experienced by the second task that

happens to be of shorter duration than the first.

5.   More Figures of Merit: Waiting Times

Continuing with a study of the properties of our process, we look

at waiting times, and the long-run rate at which tasks are completed.

(1)   Suppose that tasks are plentiful, and that each time a block ends

a task is readied for the following gap. A task of length L will be

forced to wait until a sufficient gap appears. We shall be interested

first in the time, S, that elapses until the end of the first gap that

accommodates our task, where time is measured from the beginning of the

first gap offered the task. Now

$$S = G_1' + B_1 + G_2' + B_2 + \ldots + G_{M-1}' + B_{M-1} + G_M'' \qquad (5.1)$$

where, given L, G_i' is the duration of a gap that is less than L,

and G_M'' is the duration of a gap longer than $L(G_o = B_o = 0)$, M is

the number of the first gap that is longer than L, and B_i is the

duration of the $i\underline{\text{th}}$ block. First,

$$P[M = m | L] = [\gamma(L)]^{m-1} [1-\gamma(L)] , \qquad m = 1, 2, \ldots \qquad (5.2)$$

18

and

$$E\left(e^{-sG_i^!}\Big|L\right) = \frac{\int_0^L e^{-sx}\gamma\{dx\}}{\gamma(L)} \; ; \; E(e^{-sG_M''}) = \frac{\int_L^\infty e^{-sx}\gamma\{dx\}}{1-\gamma(L)} \qquad (5.3)$$

From (5.1), then, we get

$$\varphi(s;L) = \left[E\ e^{-sS}\Big|L\right] = \frac{\int_L^\infty e^{-sx}\gamma\{dx\}}{1-\hat\beta(s)\int_0^L e^{-sx}\gamma\{dx\}} \qquad (5.4)$$

where $\hat\beta(s)$ is the L.S. transform of the block duration d.f., $\beta(x)$.
(a)   Again take $\gamma(x) = 1-e^{-vx}$.   Then

$$\int_L^\infty e^{-sx}\gamma\{dx\} = \frac{v}{v+s}\ e^{-(s+v)L}$$

and

$$\int_0^L e^{-sx}\gamma\{dx\} = \frac{v}{v+s}\ \left[1-e^{-(s+v)L}\right]$$

and

$$\varphi(s;L) = \frac{v}{[s+v(1-\hat\beta)]e^{(s+v)L}+v\hat\beta} \ . \qquad (5.5)$$

Differentiation then yields the following moments:

$$E(S|L) = \frac{1}{v}\ [1+vE(B)]e^{vL} -E(B) \ . \qquad (5.6)$$

$$E(S^2|L) = 2\ E^2(S|L) + E(B^2)(e^{vL}-1) - 2[1+vE(B)]\frac{L}{v}\ e^{vL}$$

19

The condition on L may be removed to give, for the mean,

$$E(S) = \frac{1}{\nu} [1+\nu E(B)] \; \hat{b}(-\nu) - E(B) \qquad (5.7)$$

Note that, if only a single task may pass through a gap, then the average time that elapses between the termination of two consecutive task-passing gaps is

$$E(S) + E(B) = \frac{1}{\nu} [1+\nu E(B)] \; \hat{b}(-\nu) \quad , \qquad (5.8)$$

and renewal theory allows the conclusion that the asymptotic average rate at which tasks are completed at the facility is $\nu\{[1+\nu E(B)] \; \hat{b}(-\nu)\}^{-1}$. Again this rate depends intimately upon the behavior of the task-length d.f., b, as the former is reflected by the Laplace-Stieltjes transform of the latter, evaluated for negative argument. If the latter does not converge, the corresponding mean will be infinite.

(2)   Next, let W denote the waiting-time for a task that arrives when the long-run probabilities for blocks and gaps, (3.9) and (3.10), prevail. The waiting time is the time that elapses from the instant the task appears until the time of first appearance of a gap longer than the task. The waiting time discussion here does not include delays in a queue of similar tasks; we treat this problem in the following section (3).

Distinguish the following mutually exclusive possibilities that a task may encounter:

(a)   The task finds a gap in progress, but the remaining gap duration exceeds the task duration. There is no wait in this case, and, according

20

to (4.2),

$$P[W=0] = \frac{E(G)}{E(G)+E(B)} \left\{ \frac{\int_o^\infty b(x)[1-\gamma(x)]dx}{E(G)} \right\} \qquad (5.9)$$

(b) The task arrives to find in progress a gap whose remaining duration is less than the task duration. Let T' denote the duration of the initial, insufficient, gap. Then

$$W = T' + B_1 + G_1' + B_2 + G_2' + \dots + B_{M-1} + G_{M-1}' + B_M \qquad (5.10)$$

where G_i', B_i, and M have the same meaning as in (1) above. From independence, for fixed L and T' there follows

$$E[e^{-sW}|T', L] = e^{-sT'} \frac{\hat{\beta}(s)[1-\gamma(L)]}{1-\hat{\beta}(s) \int_o^L e^{-sx}\gamma\{dx\}} \qquad (5.11)$$

Given L, the chance that a task arrives during an insufficient gap is

$$\frac{\int_o^L [1-\gamma(x)]dx}{E(G)+E(B)} \qquad (5.12)$$

and

$$\left[E\, e^{-sT'}\Big|L,\ C = 0,\ T < L\right] = \frac{\int_o^L e^{-sx}[1-\gamma(x)]dx}{\int_o^L [1-\gamma(x)]dx} \qquad (5.13)$$

Therefore we have

21

$$E[e^{-sW}|L,\ C = 0,\ T < L] = \frac{\displaystyle\int_0^L e^{-sx}[1-\gamma(x)]dx}{\displaystyle\int_0^L [1-\gamma(x)]dx} \cdot \frac{\hat{\beta}(s)[1-\gamma(L)]}{1-\hat{\beta}(s)\displaystyle\int_0^L e^{-sx}\gamma\{dx\}} \quad (5.14)$$

(c) The task finds a block in progress; T denotes the remaining duration of this block. Then

$$W = T + G_1' + B_1 + G_2' + B_2 + \ldots + G_{M-1}' + B_{M-1} \quad (5.15)$$

Use of (5.2) and (5.3) leads to

$$E[e^{-sW}|L,\ C = 1] = \frac{\displaystyle\int_0^{\infty} e^{-sx}[1-\beta(x)]dx}{E(B)} \left\{\frac{\hat{\beta}(s)[1-\gamma(L)]}{1-\hat{\beta}(s)\displaystyle\int_0^L e^{-sx}\gamma\{dx\}}\right\} \quad (5.16)$$

Finally, if all conditions save that on L are removed we obtain

$$E[e^{-sW}|L] = \frac{1}{E(G)+E(B)} \times$$

$$\left[\int_L^{\infty}[1-\gamma(x)]dx + (1-\gamma(L))\frac{\hat{\beta}(s)\displaystyle\int_0^L e^{-sx}[1-\gamma(x)]dx + \int_0^{\infty} e^{-sx}[1-\beta(x)]dx}{1-\hat{\beta}(s)\displaystyle\int_0^L e^{-sx}\gamma\{dx\}}\right] \quad (5.17)$$

when the condition on L is removed, the resulting expression is the L.S. transform derived. The expression is very complex, and it will be necessary to deal with special cases. Note that the above transform tends to unity as $s \to 0$, and is completely monotone (see Widder [3]).

Hence it is the transform of a bona-fide distribution function. However, in view of what we have seen earlier, it can be expected that the moments of the resulting d.f. will often be infinite.

Example. Take $\gamma(x) = 1 - e^{-\nu x}$. Then

$$1-\gamma(L) = 1-e^{-\nu L}, \quad \int_L^\infty [1-\gamma(x)]dx = \frac{e^{-\nu L}}{\nu}$$

$$\int_o^L e^{-sx}[1-\gamma(x)]dx = \frac{1-e^{-(s+\nu)L}}{s+\nu}, \quad \int_o^L e^{-sx}e^{-\nu x}\,\nu dx = \frac{\nu}{\nu+s}[1-e^{-(s+\nu)L}]$$

If these expressions are substituted into (5.17) the result may be simplified to give

$$E[e^{-sW}|L] = \frac{e^{-\nu L}}{1-\frac{\nu}{\nu+s}\hat{\beta}(s)[1-e^{-(s+\nu)L}]} \left\{ \frac{1+\nu\int_o^\infty e^{-sx}[1-\beta(x)]dx}{1+\nu E(B)} \right\} \quad (5.18)$$

The first moment, obtained by differentiation, is

$$E[W|L] = \frac{\nu E(B)}{1+\nu E(B)} \left\{ \frac{E(B^2)}{2E(B)} \right\} + \frac{1+\nu E(B)}{\nu}(e^{\nu L} - 1) - L \quad (5.19)$$

and removing the condition on L shows, in particular, that the expected value depends upon $E[e^{\nu L}] = \hat{b}(-\nu)$, and will be infinite if that expectation fails to exist.

(3) The discussion of the waiting time of tasks can be made to include the delays due to queueing. This effect could be the result of the tasks arriving at the facility in a random fashion, only to find the facility

23

temporarily in use by another task. Possibly then a queue would form, and each task would use the facility when his turn comes. Another possibility is that the waiting tasks are assigned facility time according to some priority scheme. We will discuss "first-come, first served" here, leaving the evaluation of priority disciplines for tasks for another time.

The complication introduced by the high-priority blocks may be treated by noting that the latter are pre-emptive interruptions in the service of the tasks. This means that tasks utilize the facility one after the other as long as their durations are shorter than the current gap. As soon as a task that is next in order at the facility finds that the remaining gap duration is shorter than his own duration, he simply postpones entering the facility until such time as a sufficient gap appears. This amounts to a virtual interruption of his service, with the task beginning from scratch when the block following the "interruption" has passed. Such virtual interruptions continue until a gap longer than the task appears.

The addition of the feature of blocks arriving to interrupt task service will tend to prolong queues of tasks. If gap durations are independently and exponentially distributed,

$$\gamma(x) = 1 - e^{-\nu x} , \qquad (5.20)$$

and tasks appear at the facility in a Poisson-wise fashion with rate λ, then the situation is one the author has examined in some detail, see Gaver[4]; the present situation may be described as one involving

24

pre-emptive - repeat interruptions, the interruptions being the blocks.
For details see [4]. We shall merely quote some results obtained in
[4] in the present context.

The assumptions already made about the gap-block process will
continue to hold, with the addition of (5.20) above. Further, we assume
that tasks appear at the facility Poisson-wise with constant rate λ.
We mention here that the methods of [4] also apply when task arrivals
are in bunches, with the bunch sizes having any distribution. For
simplicity here the bunch sizes are assumed to be unity.

It follows from [4] that

(a)   The traffic intensity parameter for the (low-priority) task queue
that forms at the facility is

$$\rho = \lambda[E(B) + \frac{1}{\nu}] \ [E(e^{\nu L}) - 1] \qquad (5.21)$$

If $\rho < 1$ the task queue fluctuates between a busy and idle condition,
and busy periods for this queue (maximal periods which one or more tasks
are present, either using the facility or waiting) terminate in finite
time with probability one and have a finite mean. Under this condition
a limiting long-run distribution for task queue length will exist.
This is the "steady state" distribution in queueing language.

If $\rho \geq 1$ a "steady-state" distribution will not exist. If
$\rho > 1$ the line will tend to grow beyond finite bounds.

It can be seen from (5.21) that a necessary condition for $\rho < 1$
is that $E(e^{\nu L}) = \hat{b}(-\nu)$ be finite. Since exactly the same condition
is required to make the expected time between individual task service

25

finite (see (5.8)), this is not surprising.

According to our present assumptions, as many tasks as possible
are processed through gaps, taking the tasks in their arrival order.
We may use (5.21) to find that the expected time between the successive
processing of tasks is

$$[E(B) + \frac{1}{\nu}] \ [E(e^{\nu L}) - 1] \tag{5.22}$$

This is to be contrasted with the result (5.8) that obtains if only one
task is processed per gap. We find, in words

$$\frac{\text{Expected time to service task, possible several tasks per gap}}{\text{Expected time to service task, one task per gap}} =$$

$$1 - [E(e^{\nu L})]^{-1} \quad ,$$

independent of the block length distribution. Of course as ν increases,
other things being equal, the advantage of "packing" gaps becomes smaller.
(b)   The __expected__ __number__ __of__ __tasks__ __in__ __the__ __system__ when the long-run distri-
bution for task queue-length prevails is, according to [4],

$$E(N) = \rho + \frac{\rho^2}{2(1-\rho)} \ \frac{E(C^2)}{[E(C)]^2} + \frac{\nu E(B)}{1+\nu E(B)} \ \frac{\lambda}{2} \ \frac{E(B^2)}{E(B)} \tag{5.23}$$

$$E(C) = \frac{\rho}{\lambda} \tag{5.24}$$

26

$$E(C^2) = 2\left[E(B) + \frac{1}{\nu}\right]^2 E\left[(e^{\nu L}-1)^2\right] +$$

$$\left[E(B^2) + \frac{2E(B)}{\nu} + \frac{2}{\nu^2}\right] \left[E(e^{\nu L}) - 1\right] \qquad (5.25)$$

$$-\left[2 E(B) + \frac{1}{\nu}\right] E\left[Le^{\nu L}\right] .$$

These come from secs. 4 and 8 of [4]; $\rho < 1$ is given by our (5.21) above.

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